INVARIANT IMBEDDING, WAVE PROPAGATION AND THE WKB APPROXIMATION

Richard Bellman
Robert Kalaba

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The RAND Corporation
1700 MAIN ST. · SANTA MONICA · CALIFORNIA

SUMMARY

In previous papers, we have presented some applications of the principle of invariant imbedding to radiative transfer and neutron diffusion processes. This use of invariance principles was stimulated by the fundamental work of Ambarzumian and Chandrasekhar, and strongly influenced by the point of regeneration technique of Bellman and Harris, and the theory of dynamic programming.

Fundamental for the success of these techniques as applied to the above processes is the ability to consider the overall physical process as a sequence of local processes. For the case of particles, this is easily done. In this paper, we wish to indicate how wave propagation may be considered in these terms. It is rather remarkable that our results will be based upon an algorithm that, in general, can yield divergent series.

Following a provocative paper by Bremmer, our aim is to show that wave propagation can be discussed in terms of reflection and refraction at infinitesimally separated interfaces. We shall prove that the convergence of the Bremmer series can be established under a simple assumption concerning the slowly varying nature of the local wave number.

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1. Introduction

In previous papers, [1,2,3,4,5], we have presented some applications of the principle of invariant imbedding to radiative transfer and neutron diffusion processes. This use of invariance principles was stimulated by the fundamental work of Ambarzumian and Chandrasekhar, [6], and strongly influenced by the point of regeneration technique of Bellman and Harris, [7], and the theory of dynamic programming, [8].

Fundamental for the success of these techniques as applied to the above processes is the ability to consider the overall physical process as a sequence of local processes. For the case of particles, this is easily done. In this paper, we wish to indicate how wave propagation may be considered in these terms. It is rather remarkable that our results will be based upon an algorithm that, in general, can yield divergent series.

Following a provocative paper by Bremmer, [9], our aim is to show that wave propagation can be discussed in terms of reflection and refraction at infinitesimally separated interfaces. We shall prove that the convergence of the Bremmer series can be established under a simple assumption concerning the slowly varying nature of the local wave number.

In addition to the insight to the physical phenomena furnished by these techniques, we are also led to a new method for studying the asymptotic behavior of ordinary and partial differential equations of linear type. For the equation

$$u'' + k^2(x)u = 0, \tag{1.1}$$

the Bremmer series is an extension of the known connection between the principal transmitted, or refracted, wave and the WKB method. We shall obtain a generalization of the WKB method to vector-matrix systems of the form

$$y'' + K^2(x)y = 0, \tag{1.2}$$

where $K(x)$ is a positive definite matrix for $x \geq 0$. This yields a new approach to the study of the asymptotic behavior of the solutions of these equations, [10].

Finally, we shall apply functional equation techniques to the problem of determining the wave reflected by an infinite half-plane of inhomogeneous material. The method we pursue is quite different from that given in Luneberg, [11].

In subsequent papers, we shall discuss the application of these ideas to general hyperbolic systems, to partial differential equations of parabolic type, and to general operator equations of the form

$$u_{tt} + A^2u = 0, \tag{1.3}$$

where A is a positive definite operator. Furthermore, we shall consider spherical and cylindrical geometries.

2. A Localization Device for Wave Processes

Consider a plane wave, $e^{i(k_0x - \omega t)}$, arriving from the homogeneous space $x < 0$ and approaching the inhomogeneous space $x > 0$. At the boundary, $x = 0$, the wave is split into a reflected wave and a refracted wave. Let us now suppose that there is an immediate reflected wave and refracted wave obtained by supposing that the inhomogeneous medium is actually homogeneous with wave number $k(+0)$. If the inhomogeneous region is now taken to be the limit of a sequence of interfaces with this type of reflection and refraction occurring at each interface, we can obtain the total reflected wave and the disturbance inside the region $x > 0$ by adding up the effects of the reflected and refracted waves obtained in this way.

It is this principle which we wish to verify under appropriate assumptions for equations (1.1) and (1.2).

3. The Bremmer Series

Let us define the sequence $\{u_n(x)\}$ where

$$u_0(x) = \left(\frac{k_0}{k(x)}\right)^{\frac{1}{2}} e^{i\int_0^x k(s)ds},$$

$$u_1(x) = -\frac{1}{2(k(x))^{1/2}} \int_x^\infty \frac{k'(s)}{k(s)^{1/2}} u_0(s) e^{i\int_x^s k(t)dt} ds,$$

$$u_{2N}(x) = \frac{1}{2(k(x))^{1/2}} \int_0^x \frac{k'(s)}{k(s)^{1/2}} u_{2N-1}(s) e^{i\int_s^x k(t)dt} ds, \tag{3.1}$$

$$u_{2N+1}(x) = -\frac{1}{2(k(x))^{1/2}} \int_x^\infty \frac{k'(s)}{k(s)^{1/2}} u_{2N}(s) e^{i\int_x^s k(t)dt} ds.$$

This series was derived by Bremmer, [9], using the localization principle of §2. Standard techniques yield

__Theorem 1. If__ $|k(x)| \geq a^2 > 0$, __for__ $x \geq 0$, __and__ $\int_0^\infty |k'(x)|dx < \infty$ __and is sufficiently small, then the Bremmer series,__ $u(x) = \sum_{n=0}^{\infty} u_n(x)$, __converges and represents a solution of__ (1.1). __Another linearly independent solution may be obtained by replacing__ i __by__ $-i$.

4. The WKB Approximation for Matrix Systems

In order to obtain an analogue of this result for matrix systems of the form of (1.2), we introduce a matrix wave-function e^{iK_0x} (we shall omit the scalar term $e^{-i\omega t}$) associated with the homogeneous space $x < 0$ possessing the wave-matrix K_0. Here e^{iK_0x} denotes the matrix exponential function; cf. [10]. At the interface, $x = 0$, between $x < 0$ and the inhomogeneous space $x > 0$, specified by e^{iK_1x}, there is a reflection and a refraction. The disturbance in $x < 0$ is given $e^{iK_0x} + e^{-iK_0x}A$, the original wave plus a reflected wave, and there is a refracted wave in $x > 0$ given by $e^{iK_1x}B$, where the matrices A and B are determined by the continuity of the functions and their derivatives at $x = 0$. Hence

$$A = (K_1 + K_0)^{-1}(K_0 - K_1), \quad B = 2(K_1 + K_0)^{-1}K_0. \tag{4.1}$$

Taking account only of the refractions, we obtain

<u>Theorem 2. The WKB approximation to (1.2) is the solution of</u>

$$\frac{dV}{dx} = \left(iK(x) - \tfrac{1}{2}K^{-1}(x)K'(x)\right)V(x), \quad V(0) = I. \tag{4.2}$$

Under the assumption that $K(x)$ is positive definite for $x \geq 0$, and that $\int_0^\infty \| K'(x) \| dx$ is sufficiently small, it can be shown that $V(x)$ is the first term of a generalized Bremmer series which converges to a solution of (1.2).

5. Reflection from an Inhomogeneous Space

Let us consider once again the scalar plane wave case. We wish to determine the reflected wave in $x < 0$ due to the inhomogeneous space $x > 0$. To obtain this, we consider the more general problem of determining the reflected wave from $x \geq z$ due to an incident wave $e^{ik_0(x-z)}$ where $k_0 = k(z - 0)$. Let the coefficient of $e^{-ik_0(x-z)}$ in $x < z$ be $u(z)$. Then applying the localization principle of §2, we see that

$$\begin{aligned} u(z) = {} & \frac{k(z)-k(z+\Delta)}{k(z)+k(z+\Delta)} + \frac{2k(z)}{k(z)+k(z+\Delta)}u(z+\Delta)\frac{2k(z+\Delta)}{k(z)+k(z+\Delta)} \\ & + \frac{2k(z)}{k(z)+k(z+\Delta)}u(z+\Delta)\frac{k(z+\Delta)-k(z)}{k(z)+k(z+\Delta)}u(z+\Delta) + O(\Delta), \end{aligned} \tag{5.1}$$

which leads to the Riccati differential equation

$$u'(z) = -\frac{k'}{k} + \frac{k'}{k}u^2. \tag{5.2}$$

If the medium is homogeneous for $z_0 < x$, with $k(x) = k_0$, then u must satisfy the boundary condition

$$u(z_0) = \frac{k(z_0-0) - k_0}{k(z_0-0) + k_0}; \tag{5.3}$$

otherwise we employ the condition $\lim_{z \to \infty} u(z) = 0$.

REFERENCES

1. Bellman, R., and R. Kalaba, "On the Principle of Invariant Imbedding and Propagation through Inhomogeneous Media," Proc. Nat. Acad. Sci. USA, Vol. 42, 1956, pp. 629–632.

2. Bellman, R., and R. Kalaba, "On the Principle of Invariant Imbedding and Diffuse Reflection from Cylindrical Regions," Proc. Nat. Acad. Sci. USA, Vol. 43, 1957, pp. 514–517.

3. Bellman, R., R. Kalaba, and G. M. Wing, "On the Principle of Invariant Imbedding and One-dimensional Neutron Multiplication," Proc. Nat. Acad. Sci. USA, Vol. 43, 1957, pp. 517–520.

4. Bellman, R., R. Kalaba, and G. M. Wing, "On the Principle of Invariant Imbedding and Neutron Transport Theory—I: One-dimensional Case," Jour. of Math. and Mech., to appear.

5. Bellman, R., R. Kalaba, and G. M. Wing, "Invariant Imbedding, Functional Equations, and Neutron Transport Theory." Jour. of Math. and Mech., to appear.

6. Chandrasekhar, S., Radiative Transfer, Oxford University Press, London, 1950.

7. Bellman, R., and T. E. Harris, "On Age-dependent Branching Processes," Proc. Nat. Acad. Sci. USA, Vol. 34, 1948, pp. 601–604.

8. Bellman, R., Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1957.

9. Bremmer, H., "The WKB Approximation as the First Term of a Geometric-optical Series," The Theory of Electromagnetic Waves, a Symposium, Interscience Publications, New York, 1951, pp. 125–138.

10. Bellman, R., Stability Theory of Differential Equations, McGraw-Hill Book Company, Inc., New York, 1954.

11. Luneberg, R. K., The Propagation of Electromagnetic Plane Waves in Plane Parallel Layers, Research Report No. 172-3, New York University, Washington Square College, June, 1947.

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